Superresolution Using Preconditioned Conjugate Gradient Method *

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ABSTRACT

In this paper we present a fast iterative image superresolution algorithm using preconditioned conjugate gradient method. To avoid explicitly computing the tolerance in the inverse filter based preconditioner scheme,[1] a new Wiener filter based preconditioner for the conjugate gradient method is proposed to speed up the convergence. The circulant-block structure of the preconditioner allows efficient implementation using Fast Fourier Transform. Effectiveness of the preconditioner is demonstrated by superresolution results for simulated image sequences.

Keywords: Superresolution, preconditioned conjugate gradient method, Wiener filter

1. INTRODUCTION

Image superresolution is the task of producing high-resolution images from a sequence of low quality, low-resolution images. Many applications, including aerial and facilities surveillance, medical, military, civilian, and scientific imaging, character recognition, etc. require high quality images. In fact, many of these vision problems suffer from inadequate resolution of the sensors, either because of cost or hardware limits. Superresolution provides an effective and economic alternative for enhancing the performance of the computer vision applications.

Superresolution, was first introduced in the literature for single-frame problems.[2] Superresolution from a single low-resolution image is known to be highly ill-posed. Recent studies mostly focus on multiframe image sequences, which can take advantage of additional spatiotemporal information available in the image sequence.[3,4] The camera and object motion allow the frames of the video sequence to contain over-sampled similar information enabling reconstruction of a superresolution image.

Superresolution algorithms can be divided into two categories: frequency domain methods and spatial domain methods. The iterative methods are the most important among the spatial domain methods, and are the focus of the work here. The main advantages of the iterative scheme include the ability to handle large image sequences, easy inclusion of spatial domain *a priori* knowledge, and the ability to handle spatially varying degradations.

There are many iterative methods to solve superresolution reconstruction problems. The Iterative Back-projection (IBP) algorithm suggested by Irani and Peleg originated from computer-aided Tomography (CAT).[5] The algorithm simulates the imaging process, backprojects the error between the simulated images and the observed low-resolution images to superresolution image, and iteratively updates it. To reduce the noise and ill-posedness, Stark et al. applied a set theoretic algorithm, projection onto convex sets (POCS), to the superresolution reconstruction.[6] It is convenient to integrate *a priori* information in POCS. However, the set theoretic algorithm suffers from non-uniqueness of the solution, slow convergence and high computational cost.

The conjugate gradient (CG) algorithm is a fundamental iterative method for effectively solving large-scale systems with very little extra storage,[7] and has been applied to the super-resolution problem.[8] The convergence rate of the standard conjugate gradient method in general is slow if the system is ill-posed. To speed up the conjugate gradient method we need to precondition the system. One of the popular preconditioning schemes is the circulant preconditioner.[9]

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In this paper we consider a simple and efficient circulant preconditioner implementation presented by Hanke and Nagy[1] for the iterative superresolution problem. This is essentially an inverse filter scheme, and it can lead to substantial speed-up. In this technique a parameter must be selected based on a knee in the L-curve (described below), and this determines the tolerance which separates the signal subspace from the noise/transient subspace. The tolerance is not easy to determine accurately. We propose a Wiener filter based preconditioner which retains the speed of the particular preconditioner, but does not require the acquisition of the tolerance explicitly.

2. THE IMAGING MODEL

The imaging process is a "forward" process that takes the ideal high-resolution image to a degraded low-resolution image. A typical imaging process consists of a geometric transformation (perhaps from 3-D to 2-D), blurring, downsampling, and an additive noise term. Using the notation of Elad and Feuer,[10] the forward imaging process can be formulated as

$$\mathbf{f}_k = DC_kF_k\mathbf{x} + \mathbf{n}_k \tag{1}$$

where

- $\mathbf{f}_k$ is the k-th low-resolution frame in the video sequence.
- D is the downsampling operator which is a known decimation operation in our case.
- C_k is the blurring operator. Knowledge of C_k for a given image sequence is usually unavailable, but it can be well approximated by a Gaussian. Studies on the CCD imaging systems show this approximation is quite reasonable and further verified by the superresolution applications.
- F_k is the geometric warping operator which maps the high-resolution grid system to the low-resolution grid systems. Motion estimation algorithms and image registration techniques can be employed to perform the geometric warping.
- $\mathbf{x}$ is the unknown ideal high-resolution image.
- $\mathbf{n}_k$ is additive noise. In superresolution, it is generally assumed that the noise is additive Gaussian with zero-mean.

The above notations are not rigidly defined, and the product of the matrices D, C_k and F_k can be interpreted in various ways depending on the forward imaging process. As we will see below the iterative method will not require its explicit construction, and we will not pursue its explicit representation. By stacking the p frame equations (1) we get

$$\begin{bmatrix}\mathbf{f}_1\\ \vdots \\ \mathbf{f}_p\end{bmatrix} = \begin{bmatrix}DC_1F_1\\ \vdots \\ DC_pF_p\end{bmatrix}\mathbf{x} + \begin{bmatrix}\mathbf{n}_1\\ \vdots \\ \mathbf{n}_p\end{bmatrix}$$

or we can rewrite it as

$$\mathbf{f} = H\mathbf{x} + \mathbf{n} \tag{2}$$

where each block of H associated with frame i has the form of block Toeplitz with "nearly" Toeplitz upper band matrix (a Toeplitz matrix is a matrix which has constant values along negative-sloping diagonals), which can be approximated by a block-Toeplitz matrix with Toeplitz-blocks.[11]

If we assume the noise is uncorrelated and with uniform variance, then maximum likelihood estimator can be used to find the solution by minimizing the function:

$$L(\mathbf{x}) = \|H\mathbf{x} - \mathbf{f}\|_2^2 \tag{3}$$

or equivalently by solving

$$H^TH\mathbf{x} = H^T\mathbf{f} \tag{4}$$

Many optimization algorithms can be utilized to solve this large sparse linear system. The conjugate gradient method is one of the favorite techniques for solving very large problems like superresolution.[12]

3. PRECONDITIONED CONJUGATE GRADIENT METHOD

Since the introduction of the conjugate gradient method by Fletcher-Reeves in the 1960s,[13] it has become a most important algorithm for large scale optimization, because it requires storage of only a few vectors, and could converge much more rapidly than the steepest descent method.

The convergence rate of the conjugate gradient method depends on the singular values of the data matrix H.[9, 14] If the singular values cluster around a fixed point, convergence will be rapid. In the case of superresolution, H is ill-conditioned with no significant gap in the singular value spectrum. To make the conjugate gradient method efficient, we need to precondition the system. The idea behind preconditioned conjugate gradients is to apply the standard conjugate gradient method to the transformed system

$$\tilde{H}^T \tilde{H} \tilde{\mathbf{x}} = \tilde{H}^T \tilde{\mathbf{f}} \tag{5}$$

where $\tilde{H} = C^{-1} H C^{-1}$, $\tilde{\mathbf{x}} = C\mathbf{x}$, $\tilde{\mathbf{f}} = C^{-1}\mathbf{f}$. Here C is a symmetric positive definite matrix which is chosen such that $\tilde{H}$ is well conditioned or a matrix with clustered eigenvalues. In practice, we define the preconditioner M by $M = C^T C$. One step of preconditioned conjugate gradient algorithm can be stated as follows[7]:

1. Compute the step length $\alpha_k = \mathbf{r}_k^T M \mathbf{r}_k / \mathbf{p}_k^T H^T H \mathbf{p}_k$.
2. Update $\mathbf{x}_k = \mathbf{x}_{k-1} + \alpha_k \mathbf{p}_{k-1}$, and the vector of residuals $\mathbf{r}_k = \mathbf{r}_{k-1} - \alpha_k H^T H \mathbf{p}_{k-1}$.
3. Compute $\beta_k = \mathbf{r}_k^T M \mathbf{r}_k / \mathbf{r}_{k-1}^T M \mathbf{r}_{k-1}^T$.
4. Update the new search direction as a combination of residual and old search direction: $p_k = M\mathbf{r}_k + \beta_k \mathbf{p}_{k-1}$.

4. WIENER FILTER BASED PRECONDITIONER

The choice of a good preconditioner can have a dramatic effect upon the rate of convergence. A variety of preconditioning schemes have been developed. In 1986, Strang[15] and Olkin[16] independently proposed using circulant matrices to precondition Toeplitz matrices, cf..[9] One of the advantages of the circulant preconditioners is that it can be easily inverted using the FFT.

One of the circulant preconditioners, originally developed by Hanke and Nagy,[1] is an inverse filter based preconditioner whose inverse can be inexpensively computed using the FFT. In their method, they first find signal subspace S by choosing a certain tolerance τ, then span all the Fourier eigenvectors corresponding to eigenvalues $|\tilde{\lambda}_j| > \tau$. Here the signal subspace S is split into two subspaces S_+ and S_-, spanned by the eigenvectors with eigenvalues $\tilde{\lambda}_j > \tau$ and $\tilde{\lambda}_j < -\tau$, respectively. The noise and transient subspace is denoted by $S^\perp$. The preconditioner M is a block circulant matrix with circulant blocks whose eigenvalues are defined as follows:

$$\mu_j = \begin{cases} 1/|\tilde{\lambda}_j|, & |\tilde{\lambda}_j| > \tau, \\ 1, & |\tilde{\lambda}_j| \leq \tau. \end{cases} \tag{6}$$

By preconditioning with M the eigenvalues larger than τ are mapped to 1, those smaller than $-\tau$ are mapped to -1, and the other eigenvalues are remaining the same values. In this way, the eigenvalues are clustered around the zero.

The unsolved problem left to implement this method is the choice of the tolerance τ. The solution suggested by Hansen and O'Leary[17] is to find a significant corner on the L-curve, which is a log-log plot of

$$\sum_{j=1}^{n-1} \frac{\eta_j^2}{\tilde{\lambda}_j^2} \qquad \text{vs.} \qquad \sum_{j=n}^{N^2} \eta_j^2.$$

where η_j is the coefficient of the decomposition of $\mathbf{f}$ on the j-th Fourier basis.

However, it is not convenient to find the tolerance in this way as it requires a human's observation. Also, there are "ringing" artifacts in the results for the inverse filter based preconditioner. To overcome the shortcomings of the inverse filter based preconditioner scheme, we extend it to the Wiener filter. The Wiener filter is a linear spatially invariant filter which tries to make the residual error as small as possible in the sense of mean squared error. The Wiener filter can be defined in the spectral domain:

$$H_{wiener} = \frac{K^*(u,v)}{|K(u,v)|^2 + S_w(u,v)/S_f(u,v)} \tag{7}$$

where $K(u,v)$ is the transfer function of the system, which can be approximated from the forward imaging process, $S_f(u,v)$ and $S_w(u,v)$ are the power spectrum of the ideal images and the noise, respectively. We can assume white noise, so $S_w(u,v) = \sigma_w^2$, which can be obtained from the degraded images. The power spectrum of the ideal image can be estimated as:

$$S_f(u,v) = S_g(u,v) - \sigma_w^2 = \frac{1}{n^2}|G(u,v)|^2 - \sigma_w^2 \tag{8}$$

The Wiener filter based preconditioner will try to cluster the eigenvalues with large absolute value to one, and also make the eigenvalues with absolute value near zero shift away from zero. The advantage of the Wiener filter based preconditioner is that it does not require one to compute the tolerance τ explicitly.

5. IMPLEMENTATION DETAILS

The main computations in the preconditioned conjugate gradient algorithm are the two matrix-vector multiplications: $H^T H\mathbf{p}$ and $M\mathbf{r}$. The former matrix-vector multiplication describes the forward then backward imaging process, and the latter is the preconditioning operation.

The matrix-vector multiplication $H^T H\mathbf{p}$ can be implemented as follows:

$\mathbf{x} = 0$;
for $i = 1$ to k **do**
  $\mathbf{x} = \mathbf{x} + \text{backproject}(\text{project}(\mathbf{p}, i), i)$;
end for

where function *project*$(\mathbf{p}, i)$ represents the forward imaging process to generate the i-th low-resolution image, and function *backproject*$(\mathbf{p}, i)$ is the backward imaging process.

Function *project*$(\mathbf{p}, i)$ will first warp the high-resolution image with the estimated motion parameters, and then blur it with the blurring operator, and decimate it to generate a low-resolution image. Function *backproject*$(\mathbf{p}, i)$ will first upsample the low-resolution image to high-resolution, then blur it with flipped blurring operator (in case of symmetric kernel, same as the original one), and backwarp it to generate a high-resolution image.

The matrix-vector multiplication $M\mathbf{r}$ can be implemented using Wiener filter scheme. Once the Wiener filter H_{wiener} is obtained from the images in the initial stage of the preconditioned conjugate gradient algorithm, we can precondition the system by multiplying it with the FFT of the residual images in the iterative stages.

6. EXPERIMENTAL RESULTS

To illustrate the effectiveness of the PCG based superresolution algorithm, we reconstruct a high-resolution images from a sequence of low-resolution images. We shift a single 256×256 image with various subpixel shifts, blur it with a 11×11 Gaussian PSF with standard deviation of 0.5, down-sample and add Gaussian noise with variance 0.002 and 0.02 to produce 64×64 low-resolution frames. From these low-resolution images, we reconstruct a superresolution image. We compare the results of our superresolution algorithm with bilinear interpolation, conjugate gradient method, and inverse filter based PCG method. Figure 1 presents the results of our superresolution algorithms on a text image sequence under noise level 0.002. Figure 2 presents the results of our superresolution algorithms on the cameraman sequence under noise level 0.002. Figure 3 shows the plots of the relative error for each iteration for text sequence and cameraman sequence under noise level 0.002 respectively, where the relative error is the F-norm of the residual errors dividing by the norm of the initial residual error. Figure 4–6 show the results under the noise level 0.02. From the results we can find that the results of conjugate gradient methods are much better bilinear interpolation method. Also we find the preconditioners can improve the convergence rate and the stability. Experiments with the larger blur show that the CG method fail to converge to the true image. The PCG methods perform much better than CG method in these cases. The conjugate gradient methods are not sensitive to the noise presented in the images sequences.

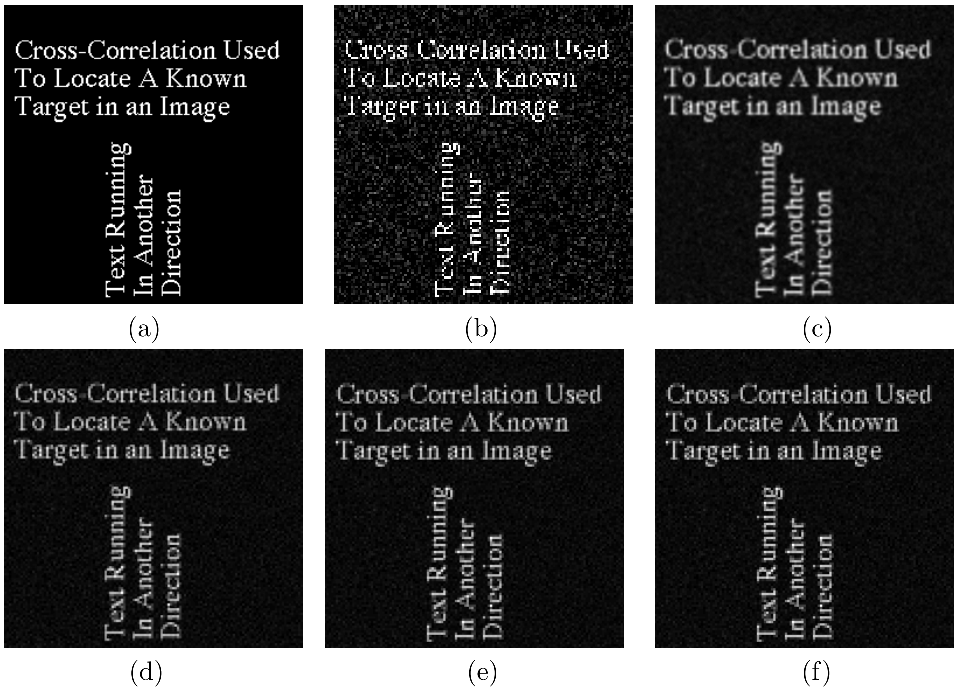

Figure 1. Superresolution results for text image sequence under noise level 0.002: (a)original image, (b)sample low-resolution image, (c)image by bilinear interpolation, (d)result by CG, (e)result by inverse filter based PCG, (f)result by Wiener filter based PCG.

7. CONCLUSIONS

We studied superresolution using the preconditioned conjugate gradient method. The discrete FFT is utilized to precondition the system quickly. Preconditioners can speed up the convergence rate and improve the stability of the CG method. We also proposed a new preconditioner based on Wiener filter scheme, which is more convenient than inverse filter based preconditioner, as there is no need to find a tolerance to separate the signal space from noise space by the way of L-curve. Experiments show that the preconditioned conjugate gradient methods make the convergence rate more rapid than that of the standard conjugate gradient method.

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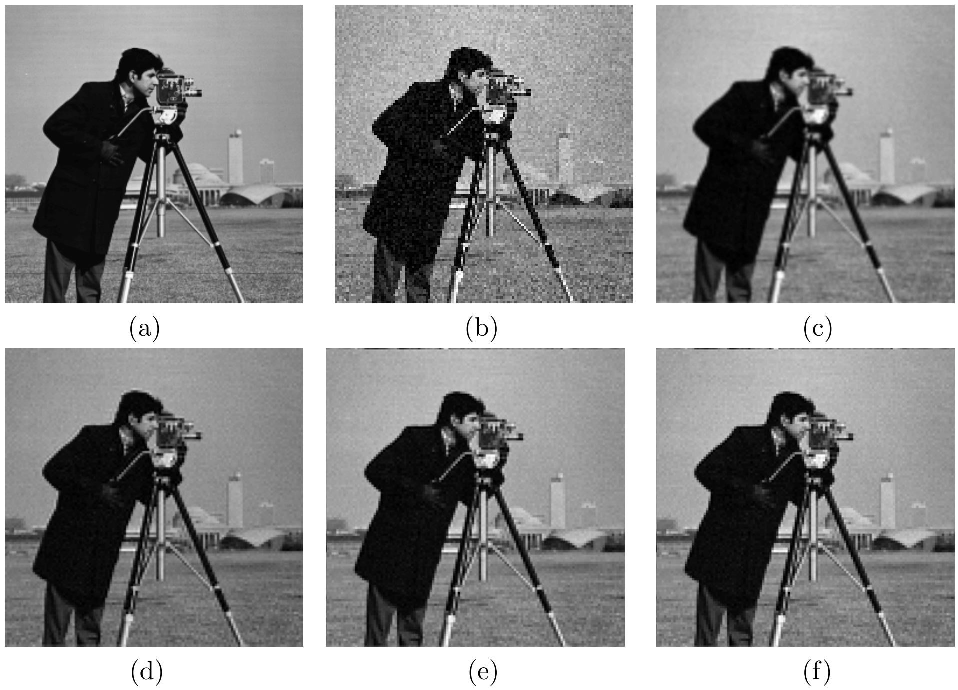

Figure 2. Superresolution results for cameraman image sequence under noise level 0.002: (a)original image, (b)sample low-resolution image, (c)image by bilinear interpolation, (d)result by CG, (e)result by inverse filter based PCG, (f)result by Wiener filter based PCG.

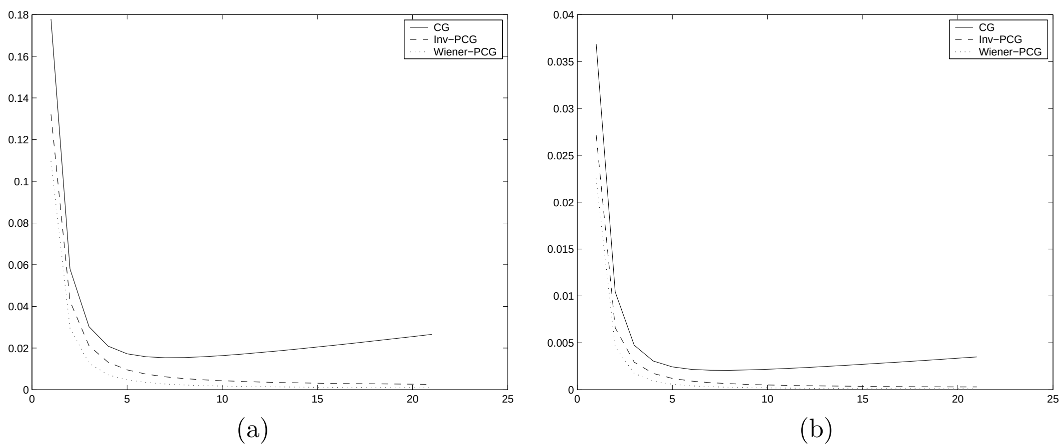

Figure 3. Convergence plot of CG and PCG for (a)text sequence and (b) cameraman sequence under noise level 0.002.

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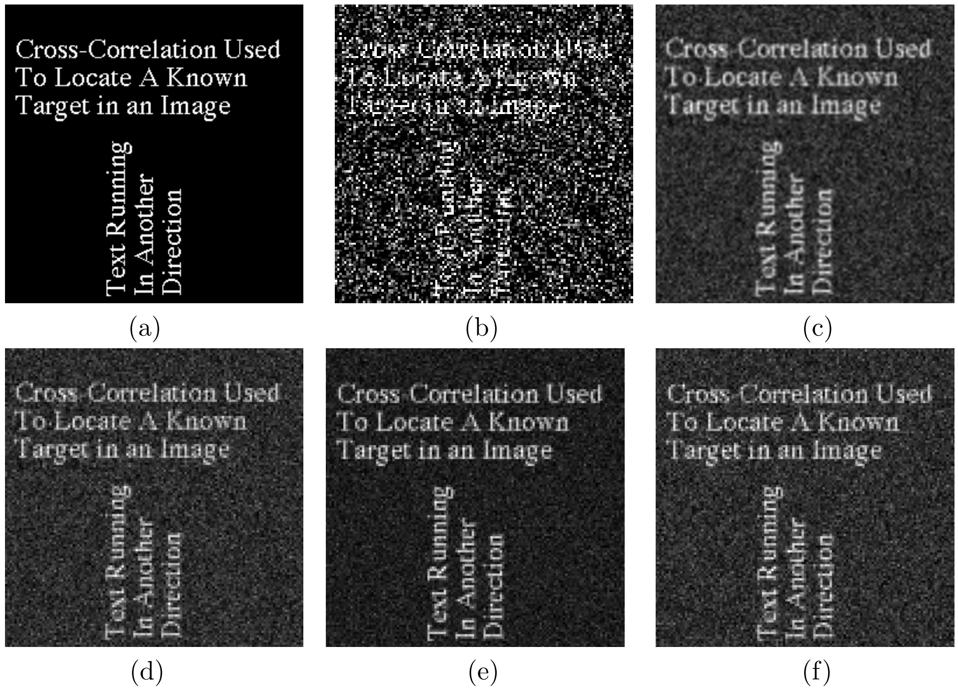

Figure 4. Superresolution results for text image sequence under noise level 0.02: (a)original image, (b)sample low-resolution image, (c)image by bilinear interpolation, (d)result by CG, (e)result by inverse filter based PCG, (f)result by Wiener filter based PCG.

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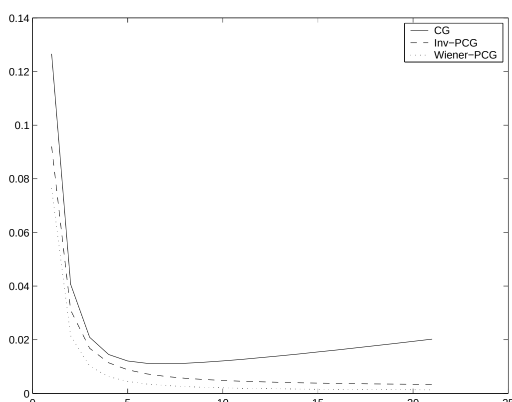
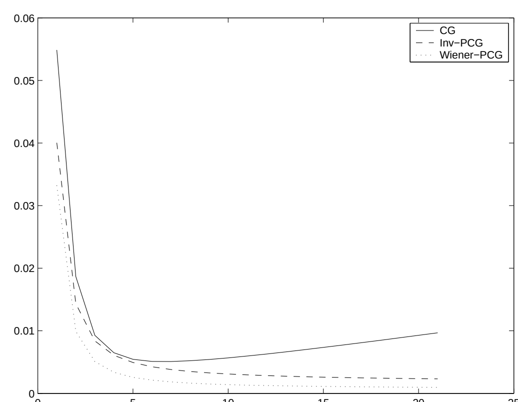
(a) (b) (c)

(d) (e) (f)

Figure 5. Superresolution results for cameraman image sequence under noise level 0.02: (a)original image, (b)sample low-resolution image, (c)image by bilinear interpolation, (d)result by CG, (e)result by inverse filter based PCG, (f)result by Wiener filter based PCG.



(a)



(b)

Figure 6. Convergence plot of CG and PCG for (a)text sequence and (b)cameraman sequence under noise level 0.02.